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THE METHOD OF PERTURBATIONS IN THE BOUNDARY VALUE PROBLEM OF THERMOVISCOELASTICITY

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We suggest and substantiate a method for the construction of the solution of the boundary value problem of thermoviscoelasticity for nonhomogeneous media in the form of a power series in the solving operator of some auxiliary homogeneous viscoelastic problem.

Considerable attention has been given recently [1 - 10] to the Boltzmann-Volterra boundary value problem of classical viscoelasticity. The problem of the uniqueness of the solution has been studied in [1, 2] and the existence of individual classes of solutions has been studied in [3 - 5]. Analytic methods for the construction of the solutions have been considered in [6 - 10] and in other papers.

1. Formulation of the problem. The complete system of equations of the quasi-statics of a viscoelastic body in the case of a given temperature field has the form [11]

$$\sigma_{ij,j} + f_i = 0, \qquad \sigma_{ij} = E^*_{ijkl}(\varepsilon_{kl} - \alpha_{kl}\theta), \qquad 2\varepsilon_{ij} = u_{i,j} + u_{j,i} \tag{1.1}$$

Here $\theta(x, t)$ is the temperature field, $f_i(x, t)$ is the volume density of the exterior forces, $u_i(x, t)$ are the quasi-static displacements, $\varepsilon_{ij}(x, t)$ is the strain tensor, $\sigma_{ij}(x, t)$ is the stress tensor, α_{ij} is the constant tensor of the anisotropic thermal dilations, $E_{ijkl}{}^*$ is the operator-tensor of anisotropic thermoviscoelasticity, x is a point in the three-dimensional Euclidean space, and t denotes the time. As usual, a repeated index indicates summation from one to three, while a comma before an index denotes differentiation with respect to the corresponding coordinate.

Making use of the symmetry of the operator-tensor

$$E^*_{ijkl} = E^*_{jikl} = E^*_{ijlk} = E^*_{klij} = E_{ijkl} - \int_0^t \Theta_{ijkl}(x, t, \tau)(\cdot)\, d\tau \tag{1.2}$$

equations (1.1) can be written in the displacements as

$$(E^*_{ijkl}u_{k,l})_{,j} + g_i = 0, \qquad g_i = f_i - \alpha_{kl}(E^*_{ijkl}\theta)_{,j} \tag{1.3}$$

We assume that the viscoelastic body occupies the bounded domain Ω. The displacements and the forces on the surface S of the body are denoted by $\varphi_i(x, t)$ and $\psi_i(x, t)$, respectively. Concerning the quasi-static conditions on the boundary S we assume that they satisfy the condition of self-conjugacy

$$\varphi_i\psi_i = 0, \qquad x \in S, \qquad 0 \leqslant t < \infty$$

The fulfilment of this condition in the case of the first, second and third fundamental boundary value problems can be achieved by introducing in the domain Ω sufficiently smooth auxiliary displacement and stress fields which remove the displacements and

forces on the boundary S or on its parts [12 – 14].
In the sequel the boundary conditions will be considered homogeneous

$$u_i|_{S_1} = 0, \qquad \sigma_{ij}n_j|_{S_2} = 0, \qquad S_1 + S_2 = S \tag{1.4}$$

Here n_j is the exterior normal vector to the surface S. We represent the problem (1.3), (1.4) in the vector form

$$V_t\mathbf{u} = \mathbf{g}, \quad x \in \Omega, \quad \mathbf{u} = 0, \quad x \in S_1, \quad \Sigma = 0, \quad x \in S_2$$
$$(V_t u)_i = -(E^*_{ijkl}u_{k,l})_{,j} \tag{1.5}$$

where V_t is the thermoviscoelasticity operator, $\Sigma_i = \sigma_{ij}n_j$ is the stress vector on the surface S. Let $\mathbf{h}$ (x, t) be a sufficiently smooth vector, satisfying the conditions (1.5) on the boundary. We multiply both sides of Eq. (1.5) by $\boldsymbol{h}$ and we integrate over the domain Ω. Since

$$\int_\Omega (\mathbf{q}\cdot V_t\mathbf{u})\,d\Omega = \int_\Omega h_{i,j}E^*_{ijkl}u_{k,l}\,d\Omega \tag{1.6}$$

we obtain the equality

$$\int_\Omega h_{i,j}E^*_{ijkl}u_{k,l}d\Omega = \int_\Omega h_i g_i d\Omega \tag{1.7}$$

The identity (1.6) defines some extension of the thermoviscoelasticity operator V_t. Therefore every vector $\mathbf{u}$ which vanishes under the conditions (1.5) on the surface S and which satisfies identically the relation (1.7) for every vector $\mathbf{h}$ from some dense set, will be called a generalized solution of the boundary value problem (1.5).

2. Reduction to an operator equation. We consider the set of the vectors M (Q_T), given in the cylinder $Q_T = \Omega \times [0, T]$, $0 \leqslant T < \infty$. We will say that the set M (Q_T) is the continuous extension in the cylinder Q_T of the functions M (Ω), given in the domain Ω, if for every vector $\mathbf{h}$ $(x, t) \in M$ (Q_T) the following conditions hold; for every fixed $t_0 \in [0, T]$ $\mathbf{h}$ $(x, t_0) \in M$ (Ω) and $\mathbf{h}$ (x, t) are continuous with respect to t in the sense of the norm on the set M (Ω). The latter allows us to introduce a norm on the set M (Q_T)

$$\|\mathbf{h}(x, t)\|_{M(Q_T)} = \sup_{0\leqslant t\leqslant T}\|\mathbf{h}(x, t)\|_{M(\Omega)} \tag{2.1}$$

The existence of the norm (2.1) follows from the continuity of the norm on M (Ω) as a function of the parameter t and from the Weierstrass theorem on the least upper bound of a continuous function. Under such a construction of the set M (Q_T), all the fundamental properties (completeness, compactness, embedding) from the set M (Ω) are carried over to it.

Let L_p (Q_T) be the continuous extension in the cylinder of the ordinary space of the vectors L_p (Ω). We consider the set M (Q_T) of the vectors which are twice differentiable in the domain Ω and which satisfy the homogeneous boundary conditions (1.4). The set M (Q_T) is dense in L_2 (Q_T), since it contains the set of finite, infinitely differentiable functions in the domain Ω which is everywhere dense in L_2 (Q_T). The closure of this set with respect to the norm generated by the inner product

$$(\mathbf{u}\cdot\mathbf{v})_{H^\circ(Q_T)} = \sup_{0\leqslant t\leqslant T}(\mathbf{u}\cdot\mathbf{v})_{H^\circ(\Omega)} = \sup_{0\leqslant t\leqslant T}\int_\Omega u_{i,j}v_{i,j}d\Omega$$

determines a Hilbert space H° (Q_T). It follows from the embedding theorem [15] that

$H^\circ(Q_T) \subset L_p(Q_T)$ and for every element $\mathbf{u} \in H^\circ(Q_T)$ we have

$$\|\mathbf{u}\|_{L_p(Q_T)} \leqslant c_p \|\mathbf{u}\|_{H^\circ(Q_T)} \qquad (1 < p \leqslant 6) \tag{2.2}$$

The constant c_p depends also on the domain Ω.

First we assume that the operator-tensor E_{ijkl}^* is positive definite. This means that for every symmetric tensor γ_{ij} we have the inequality

$$E_{ijkl}\gamma_{ij}\gamma_{kl} > c\gamma_{ij}\gamma_{ij} \tag{2.3}$$

where the positive constant c does not depend on the tensor E_{ijkl} and the point x. The inequality (2.3) allows us to introduce on the set $M(Q_T)$ the inner product

$$(\mathbf{u}\cdot\mathbf{v})_{H(Q_T)} = \sup_{0\leqslant t\leqslant T} (\mathbf{u}\cdot\mathbf{v})_{H(\Omega)} = \sup_{0\leqslant t\leqslant T} \int_\Omega E_{ijkl} u_{i,j} v_{k,l} d\Omega \tag{2.4}$$

which, by virtue of the symmetry of the tensor E_{ijkl} is a bilinear form of the elastic deformation energy. The closure of the set $M(Q_T)$ with respect to the norm generated by the inner product (2.4) leads to the Hilbert space $H(Q_T)$. The space $H(Q_T)$ is equivalent to the space $H^\circ(Q_T)$ in the sense that they consist of the same elements and that from the convergence in one space the convergence in the other space follows. Consequently, $H(Q_T) \subset L_p(Q_T)$. By virtue of (2.2) and Korn's inequality [13], we have

$$\|\mathbf{u}\|_{H^\circ(Q_T)} \leqslant c\|\mathbf{u}\|_{H(Q_T)}, \qquad \|\mathbf{u}\|_{L_p(Q_T)} \leqslant c_p \|\mathbf{u}\|_{H(Q_T)} \tag{2.5}$$

Obviously, the constants in (2.5) and (2.2) are distinct.

We introduce the operator A_t such that for a fixed $t \in [0, T]$ for all vectors $\mathbf{u}$, $\mathbf{h} \in H(Q_T)$ we have the identity

$$(\mathbf{h}\cdot A_t\mathbf{u})_{H(\Omega)} = \int_\Omega h_{i,j} \int_0^t \vartheta_{ijkl}(x, t, \tau)\, u_{k,l}\, d\tau d\Omega \tag{2.6}$$

Then the fundamental equality (1.7) can be represented as:

$$(\mathbf{h}\cdot\mathbf{u})_{H(\Omega)} - \mu(\mathbf{h}\cdot A_t\mathbf{u})_{H(\Omega)} = (\mathbf{h}\cdot\mathbf{g}) \tag{2.7}$$

Here $(\mathbf{h}\cdot\mathbf{g})$ is the usual inner product of the vectors with respect to the domain Ω, μ is the perturbation parameter. For $\mu = 1$ we obtain (1.7).

For every fixed vector $\mathbf{g} \in L_q(Q_T)$, $^6/_5 \leqslant q$ the inner product $(\mathbf{h}\cdot\mathbf{g})$ defines a linear functional on $H(Q_T)$. Indeed, for every vector $\mathbf{h} \in H(Q_T)$ from Hölder's inequality [15] and from the estimates (2.5) we have

$$|(\mathbf{h}\cdot\mathbf{g})| \leqslant \|\mathbf{h}\|_{L_p(\Omega)} \|\mathbf{g}\|_{L_q(\Omega)} \leqslant c_p \|\mathbf{g}\|_{L_q(\Omega)} \|\mathbf{h}\|_{H(\Omega)} \leqslant c\|\mathbf{h}\|_{H(\Omega)} \tag{2.8}$$

i.e. for the selected values of q the space $H(\Omega)$ is embedded in $L_p(\Omega)$ $(1 < p \leqslant 6)$. By virtue of Riesz' theorem [16] there exists a unique element $\mathbf{v} \in H(Q_T)$, such that

$$(\mathbf{h}\cdot\mathbf{g}) = (\mathbf{h}\cdot\mathbf{v})_{H(\Omega)} \tag{2.9}$$

This gives the possibility to represent the equality (2.7) in the form

$$([\mathbf{u} - \mu A_t\mathbf{u} - \mathbf{v}]\cdot\mathbf{h})_{H(\Omega)} = 0$$

for every $\mathbf{h} \in H(Q_T)$. rom here it follows that

$$\mathbf{u} - \mu A_t\mathbf{u} = \mathbf{v} \tag{2.10}$$

The right-hand side of the operator equation (2.10) is the solution of the elastic problem.

Indeed, substituting in (2.7) $\mu = 0$, we obtain

$$(\mathbf{h}\cdot\mathbf{u})_{H(\Omega)} = (\mathbf{h}\cdot\mathbf{g}) \tag{2.11}$$

From the uniqueness of the representation (2.9) it follows that $\mathbf{v} = \mathbf{u}$. But equality (2.11) determines [13] the inverse operator of the elastic problem

$$\mathbf{u} = E^{-1}\mathbf{g}, \qquad (E\mathbf{u})_i = -E_{ijkl}u_{k,l} \tag{2.12}$$

It is easy to verify that this operator is selfadjoint and bounded

$$\| E^{-1}\mathbf{g} \|_{H(Q_T)} \leqslant c_p \| \mathbf{g} \|_{L_q(Q_T)}, \qquad q \geqslant {}^6/_5 \tag{2.13}$$

As a result the operator equation (2.10) is represented in the form

$$\mathbf{u} - \mu A_t \mathbf{u} = E^{-1}\mathbf{g} \tag{2.14}$$

The following reduces to the investigation of the properties of this equation.

3. Construction of the solution. Aside from the condition (2.3) on the operator tensor E_{ijkl}^* we impose an additional requirement: for every fixed values $0 \leqslant t,\ \tau \leqslant T$ the kernel $\partial_{ijkl}(x, t, \tau) \in C(\Omega)$; in addition

$$\| \partial_{ijkl}(x, t', \tau') - \partial_{ijkl}(x, t, \tau) \|_{C(\Omega)} \to 0, \quad t' \to t, \quad \tau' \to \tau \tag{3.1}$$

Here $C(\Omega)$ is the space of continuous functions.

Lemma 1. The operator A_t acts in the space $H(Q_T)$ and we have the estimate

$$\| A_t \mathbf{u} \|_{H(\Omega)} \leqslant \int_0^t \Gamma(t, \tau) \| \mathbf{u} \|_{H(\Omega)}(\tau)\, d\tau \tag{3.2}$$

where $\Gamma(t, \tau)$ is a continuous function, connected with the tensor ∂_{ijkl} and the domain Ω.

Proof. We evaluate the norm of the operator A_t. To this end we choose $\mathbf{h} = A_t\mathbf{u}$ and we make use of the identity (2.6). This gives

$$\| A_t \mathbf{u} \|^2_{H(\Omega)} = \left| \left((A_t\mathbf{u})_{i,j} \cdot \int_0^t \partial_{ijkl}(x, t, \tau)\, u_{k,l}(x, \tau)\, d\tau \right) \right| \tag{3.3}$$

From Buniakowski's inequality we obtain

$$\| A_t \mathbf{u} \|^2_{H(\Omega)} \leqslant \| (A_t\mathbf{u})_{i,j} \|_{L_2(\Omega)} \left\| \int_0^t \partial_{ijkl}(x, t, \tau)\, u_{k,l}\, d\tau \right\|_{L_2(\Omega)} \tag{3.4}$$

From the definition of the norm and the estimate (2.5) we have

$$\| (A_t \mathbf{u})_{i,j} \|_{L_2(\Omega)} \leqslant \| A_t \mathbf{u} \|_{H^0(\Omega)} \leqslant c \| A_t \mathbf{u} \|_{H(\Omega)} \quad (i, j = 1, 2, 3) \tag{3.5}$$

Estimating the second factor in (3.4) with the aid of Minkowski's inequality [15], we find

$$\left\| \int_0^t \partial_{ijkl}(x, t, \tau)\, u_{k,l}\, d\tau \right\|_{L_2(\Omega)} \leqslant c \int_0^t K_{ij}(\tau, \tau) \| \mathbf{u} \|_{H(\Omega)}(\tau)\, d\tau \tag{3.6}$$

$$K_{ij}(t, \tau) = \sum_{k,l=1}^{3} \| \partial_{ijkl}(x, t, \tau) \|_{C(\Omega)}$$

Making use of the inequalities (3.5), (3.6) from (3.4), we obtain the estimate (3.2), where

$$\Gamma(t, \tau) = c \sum_{i,j=1}^{3} K_{ij}(t, \tau) \tag{3.7}$$

Now it is easy to prove the strong continuity of the operator A_t with respect to t We have

$$([A_{t'}\mathbf{u} - A_t\mathbf{u}]\cdot\mathbf{h})_{H(\Omega)} = \left(\frac{t'}{t} - 1\right)(A_t\mathbf{n}\cdot\mathbf{h})_{H(\Omega)} + \frac{t'}{t}\,((J_1 + J_2 + J_3)\mathbf{u}\cdot\mathbf{h})_{H(\Omega)} \tag{3.8}$$

Here the following notation has been introduced for the operators

$$(J_1\mathbf{u}\cdot\mathbf{h})_{H(\Omega)} = \int_\Omega h_{i,j}\int_0^t \partial_{ijkl}(x, t, \tau)\left[U_{k,l}\left(x, \frac{t'}{t}\tau\right) - u_{k,l}(x, \tau)\right] d\tau\, d\Omega$$

$$(J_2\mathbf{u}\cdot\mathbf{h})_{H(\Omega)} - \int_\Omega h_{i,j}\int_0^t\left[\partial_{ijkl}\left(x, t', \frac{t'}{t}\tau\right) - \partial_{ijkl}(x, t, \tau)\right]u_{k,l}(x, \tau)\, dt\, d\Omega \tag{3.9}$$

$$(J_3\mathbf{u}\cdot\mathbf{h})_{H(\Omega)} = \int_\Omega h_{i,j}\int_0^t\left[\partial_{ijkl}\left(x, t', \frac{t'}{t}\tau\right) - \partial_{ijkl}(x, t, \tau)\right]\times$$
$$\times\left[u_{k,l}\left(x, \frac{t'}{t}\tau\right) - u_{k,l}(x, \tau)\right] d\tau\, d\Omega$$

The relation (3.8) is obtained through the change of variable $\tau' = (t'/t)\tau$ in the operator A_t and identical transformations. For $t' \to t$ the first term in (3.8) tends to zero. Let us show that the same holds for the other terms too. It is clear that

$$(J_1\mathbf{u}\cdot\mathbf{h})_{H(\Omega)} = \left(A_t\left[\mathbf{u}\left(x, \frac{t'}{t}\tau\right) - \mathbf{u}(x, \tau)\right]\cdot\mathbf{h}\right)_{H(\Omega)} \tag{3.10}$$

Making use of the Buniakowski's inequality and then taking into account (3.2), we obtain

$$|(J_1\mathbf{u}\cdot\mathbf{h})_{H(\Omega)}| \leqslant \left\|A_t\left[\mathbf{u}\left(x, \frac{t'}{t}\tau\right) - \mathbf{u}(x, \tau)\right]\right\|_{H(\Omega)}\|\mathbf{h}\|_{H(\Omega)} \leqslant \tag{3.11}$$
$$\leqslant \|\mathbf{h}\|_{H(\Omega)}\int_0^t \Gamma(t, \tau)\left\|\mathbf{u}\left(x, \frac{t'}{t}\tau\right) - \mathbf{u}(x, \tau)\right\|_{H(\Omega)} d\tau \to 0, \qquad t' \to t$$

since $\mathbf{u}(x, t) \in H(Q_T)$ by assumption and $\|\mathbf{h}\|_{H(\Omega)}$ does not depend on t'. For the estimate of the second expression of (3.9) we apply Hölder's inequality with exponents $p = q = 2$. Proceeding in the same way as we have done for the estimate (3.6), we obtain

$$|(J_2\mathbf{u}\cdot\mathbf{h})_{H(\Omega)}| \leqslant \|h_{i,j}\|_{L_2(\Omega)}\left\|\int_0^t\left[\partial_{ijkl}\left(x, t', \frac{t'}{t}\tau\right) - \partial_{ijkl}(x, t, \tau)\right]u_{k,l}d\tau\right\|_{L_2(\Omega)} \leqslant$$
$$\leqslant \|\mathbf{h}\|_{H(\Omega)}\sum_{i,j=1}^{3}\int_0^t\left\|\partial_{ijkl}\left(x, t', \frac{t'}{t}\tau\right) - \partial_{ijkl}(x, t, \tau)\right\|_{C(\Omega)}\|\mathbf{u}\|_{H(\Omega)}(\tau)\, d\tau \to 0, \qquad t' \to t \tag{3.12}$$

since the condition (3.1) holds. From (3.11) and (3.12) it follows that we also have

$$|(J_3\mathbf{u}\cdot\mathbf{h})_{H(\Omega)}| \to 0, \qquad t' \to t \tag{3.13}$$

The lemma is proved.

L e m m a 2. The operator A_t is a Volterra operator in the space $H(Q_T)$.

P r o o f. From Lemma 1 and from Riesz' compactness criterion [16] it follows that the operator A_t is completely continuous in $H(Q_T)$. It remains to prove [17] that the resolvent $(\lambda I - A_t)$, $\lambda = \mu^{-1}$ exists for all finite values of the parameter λ, except for $\lambda = 0$. We have

$$(\lambda I - A_t)^{-1}\mathbf{h} = \lambda^{-1}\sum_{n=0}^{\infty}\lambda^{-n}A_t^n\mathbf{h} \tag{3.14}$$

Since A_t acts in $H(Q_T)$, its power will also act in this space. In addition, we have the estimate

$$\| A_t^n\mathbf{h}\|_{H(\Omega)} \leqslant \int_0^t \Gamma_{n-1}(t,\tau)\,\|\mathbf{h}\|_{H(\Omega)}\,d\tau, \qquad \mathbf{h}\in H(Q_T) \tag{3.15}$$

where $\Gamma_{n-1}(t,\tau)$ is the $(n-1)$-st iterate of the kernel (3.7). Indeed, by induction

$$\| A_t^{n+1}\mathbf{h}\|_{H(\Omega)} = \| A_t(A_t^n\mathbf{h})\|_{H(\Omega)} \leqslant \int_0^t \Gamma(t,\tau)\,\| A_\tau^n\mathbf{h}\|_{H(\Omega)}\,d\tau \leqslant \tag{3.16}$$

$$\leqslant \int_0^t \Gamma(t,\tau)\,d\tau\int_0^t \Gamma_{n-1}(\tau,\theta)\,\|\mathbf{h}\|_{H(\Omega)}\,d\theta = \int_0^t \|\mathbf{h}\|_{H(\Omega)}\,d\theta\int_\theta^t \Gamma(t,\tau)\,\Gamma_{n-1}(\tau,\theta)\,d\tau$$

The last integral in (3.16) is the n-th iterate of the kernel $\Gamma(t,\tau)$. Making use of the estimate (3.15), we obtain

$$\Big\| \sum_{n=0}^{\infty}\lambda^{-n}A_t^n\mathbf{h}\|_{H(\Omega)} \leqslant \Big(I + \sum_{n=1}^{\infty}|\lambda|^{-n}\int_0^t \Gamma_{n-1}(t,\tau)(\cdot)\,d\tau\,\|\mathbf{h}\|_{H(\Omega)} \tag{3.17}$$

i.e. the series for the resolvent is strongly majorized by the Neumann series for the usual Volterra operator, which, as is known, converges for every finite value of the parameter $\lambda^{-1} = \mu$. The lemma is proved.

Theorem 1. If the temperature field $\theta(x,t) \in W_p^{(1)}(Q_T)$ $(p > 3)$ and the force-vector $\mathbf{f}(x,t) \in L_q(Q_T)$ $(q \geqslant {}^6/_5)$, then there exists a unique solution $\mathbf{u}(x,t) \in H(Q_T)$, of the thermoviscoelasticity problem which can be represented by the series

$$\mathbf{u} = \sum_{n=0}^{\infty} A_t^n E^{-1}\mathbf{g} \tag{3.18}$$

Proof. The operator tensor E^*_{ijkl} is an analytic function of the temperature [11], therefore its properties in the domain Ω are determined entirely by the temperature field (only in the case of temperature nonhomogeneity). By assumption, the temperature field is Sobolev differentiable, and the derivatives are summable with exponent $p>3$. From the embedding theorem [15] it follows that in this case the temperature $\theta(x,t) \in C(Q_T)$ i.e. it is a continuous function. Consequently, the operator tensor E^*_{ijkl} satisfies the additional requirement (3.1). The right-hand side of (1.3) $\mathbf{g}(x,t) \in L_q(Q_T)$ $(q \geqslant {}^6/_5)$. Indeed, $\mathbf{f}(x,t) \in L_q(Q_T)$ $(q \geqslant {}^6/_5)$ by assumption. The remaining terms contain derivatives of the temperature, which are summable with higher exponents.

Thus, the conditions of the Theorem ensure the equivalence of the boundary value problem (1.5) of thermoviscoelasticity (in its generalized formulation (1.7)) with the operator equation (2.14) and also the solvability of the latter. Making use of Lemma 2, for $\mu = 1$ we obtain the series (3.18). From the inequalities (3.17) and (2.3) we obtain the estimate of the solution

$$\|\mathbf{u}\|_{H(Q_T)} \leqslant c_T'\,\|E^{-1}\mathbf{g}\|_{H(Q_T)} \tag{3.19}$$

where the constant c_T' is expressed in terms of the norms in $C(Q_T)$ of the kernels $Э_{ijkl}$. This can be easily verified on the basis of the equalities (3.6) and (3.9).

$$c_T' = \exp\left(T \sup_{0\leqslant t,\, \tau\leqslant T} \Gamma(t, \tau)\right) = \exp\left(cT_{ijkl} \max \| \vartheta_{ijkl}(x, t, \tau) \|_{C(Q_T)}\right) \quad (3.20)$$

Making use of the boundedness of the inverse operator of the elastic problem (2.13), we obtain

$$\| \mathbf{u} \|_{H(Q_T)} \leqslant c_T \| \mathbf{g} \|_{L_q(Q_T)}, \qquad q \geqslant {}^6/_5 \quad (3.21)$$

Note. In the investigation of the second boundary value problem it is necessary to take into account that the initial set of functions $M(Q_T)$ must satisfy the additional kinematic conditions

$$\int_\Omega \mathbf{u} d\Omega = 0, \qquad \int_\Omega [\mathbf{r} \times \mathbf{u}]\, d\Omega = 0 \quad (3.22)$$

where $\mathbf{r}$ is the radius-vector of the point in the body. Furthermore, relative to the domain Ω aside from the usual "cone condition" it is necessary to make some additional assumptions [13].

4. On the efficiency of the solution. The construction of the solution (3.18) is based on the preliminary inversion of the operator of the elastic problem. The estimate (3.19), (3.20) show that the deviation from the elastic solution is larger if the time T and the norm of the tensor ϑ_{ijkl} are larger. Practically this means that we will have an acceptable convergence of the series (3.18) only if the norm of the operators A_t is small. Thus, there arises the problem of the efficiency of the solution.

An improvement of the convergence can be obtained by a modification of the construction of the solution. The operator tensor (1.2) is represented in the form

$$E_{ijkl}^* = \overset{\circ}{E}_{ijkl} - \Delta_{ijkl}^* \quad (4.1)$$

$$\overset{\circ}{E}_{ijkl} = E_{ijkl} - \int_0^t \overset{\circ}{\vartheta}_{ijkl}(t, \tau)(\cdot)\, d\tau, \quad \Delta_{ijkl}^* = \int_0^t [\vartheta_{ijkl}(x, t, \tau) - \overset{\circ}{\vartheta}_{ijkl}(t, \tau)]\,(\cdot)\, d\tau$$

The homogeneous kernel $\overset{\circ}{\vartheta}_{ijkl}$ is selected conveniently. In particular, we can take the average

$$\overset{\circ}{\vartheta}_{ijkl}(t, \tau) = \frac{1}{|\Omega|} \int_\Omega \vartheta_{ijkl}(x, t, \tau)\, d\Omega$$

or the value of the kernel ϑ_{ijkl} at some characteristic point of the domain $\Omega + S$. Then the thermoviscoelastic operator is represented in the form

$$V_t = V_t^\circ - \Delta_t \quad (V_t^\circ(\cdot))_i = \overset{\circ}{E}_{ijkl}(\cdot)_{k,lj}, \quad (\Delta_t(\cdot))_i = (\Delta_{ijkl}^*(\cdot)_{k,l})_{,j} \quad (4.2)$$

in accordance with the selected problem of homogeneous viscoelasticity. The extension of the operators V_t° and Δ_t to the set $H(Q_T)$ leads to the integral identities

$$(\mathbf{h} \cdot V_t^\circ \mathbf{u})_{H(\Omega)} = (\mathbf{h} \cdot [\mathbf{u} - A_t^\circ \mathbf{u}])_{H(\Omega)},$$

$$(\mathbf{h} \cdot A_t^\circ \mathbf{u})_{H(\Omega)} = \int_\Omega h_{i,j} \int_0^t \overset{\circ}{\vartheta}_{ijkl}(t, \tau)\, u_{k,l}\, d\tau d\Omega \quad (4.3)$$

$$(\mathbf{h} \cdot \Delta_t \mathbf{u})_{H(\Omega)} = \int_\Omega h_{i,j} \int_0^t [\vartheta_{ijkl}(x, t, \tau) - \overset{\circ}{\vartheta}_{ijkl}(t, \tau)]\, u_{k,l}\, d\tau d\Omega$$

In the same way as before (see Sect. 2), the thermoviscoelasticity problem reduces to the operator equation

$$\mathbf{u} - A_t^\circ \mathbf{u} - \Delta_t \mathbf{u} = E^{-1} \mathbf{g} \quad (4.4)$$

The operator $I - A_t^\circ$ admits an inverse since it is a special case of the operator

$I - A_t$. Therefore Eq. (4.4) can be written as:

$$\mathbf{u} - B_t\mathbf{u} = V_t^{\circ -1}\mathbf{g}$$
$$B_t = (I - A_t^\circ)^{-1}\Delta_t, \qquad V_t^{\circ -1} = (I - A_t^\circ)^{-1} E^{-1} \tag{4.5}$$

where $V_t^{\circ -1}$ is the inverse of the operator of the homogeneous viscoelastic problem. Based on the previous results we can verify that the operator B_t acts in $H\ (Q_T)$ and is a Volterra operator in this space.

Theorem 2. Under the conditions of Theorem 1 we represent the solution of the thermoviscoelasticity problem in the form

$$\mathbf{u} = \sum_{n=0}^{\infty} B_t^n V_t^{\circ -1}\mathbf{g} \tag{4.6}$$

The improved convergence of the series (4.6) in comparison with (3.18) is connected with the fact that by an appropriate choice of the tensor $\overset{\circ}{\partial}_{ijkl}$ the norm of the operator B_t can be made sufficiently small, in particular when the tensor ∂_{ijkl} is weakly nonhomogeneous

$$\| B_t \| \leqslant \| \Delta_t \| \| (I - A_t^\circ)^{-1} \| \leqslant c_T \max_{ijkl} \| \partial_{ijkl}(x, t, \tau) - \partial^\circ_{ijkl}(t, \tau) \|_{C(Q_T)} \times$$
$$\times \exp\left(cT \max_{ijkl} \sup_{0 \leqslant t,\, \tau \leqslant T} | \overset{\circ}{\partial}_{ijkl}(t, \tau) |\right) \tag{4.7}$$

For the derivation of (4.7) we have made use of the estimates for the operators Δ_t and $V_t^{\circ -1}$ similar to that of (3.2) and (3.19).

Thus, the process of constructing the solution of the boundary value problem of thermoviscoelasticity reduces to the solution of some homogeneous viscoelastic problem with a subsequent computation of corrections due to the nonhomogeneity. The efficiency of the method will depend in a large degree from the selected homogeneous problem. In many cases, the solution of the homogeneous problem can be constructed elementarily, based on the well developed algebra of Volterra operators or on the methods of operational calculus.

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ON THE IMBEDDING OF A THIN RIGID BODY IN A PLASTIC MEDIUM WITH HARDENING

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The problem of the imbedding of a solid, thin, well-lubricated cutting edge in a half-space of rigidly plastic hardening material under plane strain conditions is considered in a linear formulation. It is assumed that translational hardening [1] occurs. The problem turns out to be kinematically determinate.

Directing the coordinate axes as shown in Fig. 1, a, let us write the equation of the cutting edge surface as

$$y = \delta f(x) \tag{1}$$

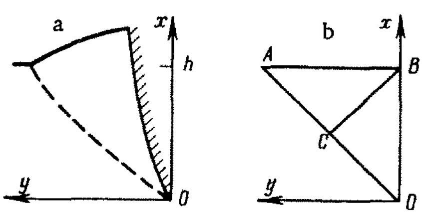

Fig. 1.

where δ is a small dimensionless parameter, and f is a sufficiently smooth function. At the initial instant the material occupies the half-space $x \leqslant 0$. Reversing the motion, let us consider the cutting edge fixed, and the medium to be displaced progressively